

1.

$$\begin{aligned}\int_{x_0}^{x_3} f(x) dx &= \frac{3\Delta x}{8} (f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)) \\&= \int_0^{3h} ax^3 + bx^2 + cx + d dx \\&= \left[\frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx \right]_0^{3h} \\&= \left(\frac{a(3h)^4}{4} + \frac{b(3h)^3}{3} + \frac{c(3h)^2}{2} + d(3h) \right) - \left(\frac{a(0)^4}{4} + \frac{b(0)^3}{3} + \frac{c(0)^2}{2} + d(0) \right) \\&= \frac{81ah^4}{4} + 9bh^3 + \frac{9ch^2}{2} + 3dh\end{aligned}$$

(LHS)

$$\begin{aligned}\frac{3\Delta x}{8} (f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)) \\&= \frac{3h}{8} (f(0) + 3f(h) + 3f(2h) + f(3h)) \\&= \frac{3h}{8} [(a(0)^3 + b(0)^2 + c(0) + d) + 3(a(h)^3 + b(h)^2 + c(h) + d) \\&\quad + 3(a(2h)^3 + b(2h)^2 + c(2h) + d) + (a(3h)^3 + b(3h)^2 + c(3h) + d)] \\&= \frac{3h}{8} [(d) + 3(ah^3 + bh^2 + ch + d) + 3(8ah^3 + 4bh^2 + 2ch + d) \\&\quad + (27ah^3 + 9bh^2 + 3ch + d)] \\&= \frac{3h}{8} [(d) + 3(ah^3 + bh^2 + ch + d) + 3(8ah^3 + 4bh^2 + 2ch + d) \\&\quad + (27ah^3 + 9bh^2 + 3ch + d)] \\&= \frac{3h}{8} [(d) + (3ah^3 + 3bh^2 + 3ch + 3d) + (24ah^3 + 12bh^2 + 6ch + 3d) \\&\quad + (27ah^3 + 9bh^2 + 3ch + d)] \\&= \frac{3h}{8} (54ah^3 + 24bh^2 + 15ch + 8d) \\&= \frac{81ah^4}{4} + 9bh^3 + \frac{9ch^2}{2} + 3dh\end{aligned}$$

(RHS)

LHS $\equiv$ RHS

QED

2.

For $\mathbb{S}_6$, $n = 6$:

$$\Delta x = \frac{7-1}{6} = 1$$

For the 1st cubic approximation $[1, 4]$

$$x_0 = 1, y_0 = 1$$

$$x_1 = 2, y_1 = 2$$

$$x_2 = 3, y_2 = 4$$

$$x_3 = 4, y_3 = 4$$

$$\int_1^4 f(x) dx \approx \frac{3\Delta x}{8} (f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3))$$

$$= \frac{3\Delta x}{8} (y_0 + 3y_1 + 3y_2 + y_3)$$

$$= \frac{3(1)}{8} (1 + 3(2) + 3(4) + 4)$$

$$= 8.625$$

For the 2nd cubic approximation $[4, 7]$

$$x_3 = 4, y_3 = 4$$

$$x_4 = 5, y_4 = 0$$

$$x_5 = 6, y_5 = 2$$

$$x_6 = 7, y_6 = 2$$

$$\int_4^7 f(x) dx \approx \frac{3\Delta x}{8} (f(x_3) + 3f(x_4) + 3f(x_5) + f(x_6))$$

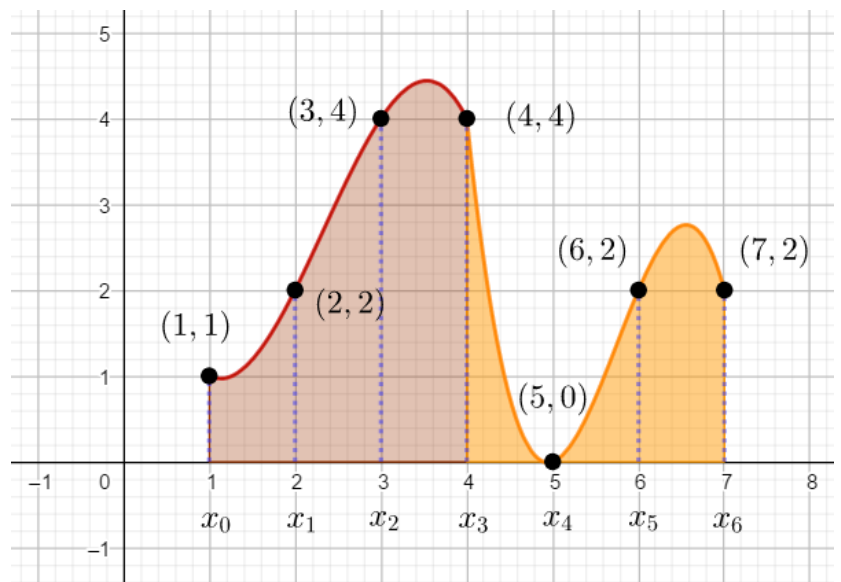
$$= \frac{3\Delta x}{8} (y_3 + 3y_4 + 3y_5 + y_6)$$

$$= \frac{3(1)}{8} (4 + 3(0) + 3(2) + 2)$$

$$= 4.5$$

$$\int_1^7 f(x) dx = \int_1^4 f(x) dx + \int_4^7 f(x) dx$$

$$\int_1^7 f(x) dx = 8.625 + 4.5 = \mathbf{13.125}$$



3.

$$f(x) = \sin x - x^5$$

$$f'(x) = \cos x - 5x^4$$

$$f''(x) = -\sin x - 20x^3$$

$$f'''(x) = -\cos x - 60x^2$$

$$f^{(4)}(x) = \sin x - 120x$$

$$f^{(5)}(x) = \cos x - 120$$

For a stationary point of $f^{(4)}(x)$ on $[0,1]$:

$$\cos x - 120 = 0$$

$$\cos x = 120$$

No solution

$$f^{(4)}(0) = \sin 0 - 120(0) = 0$$

$$f^{(4)}(1) = \sin 1 - 120(1) = -119.16$$

$$\text{On } [0,1], -120 < f^{(4)}(x) \leq 0$$

$$|f^{(4)}(x)| < 120$$

$$|E_{\mathbb{S}_n}| < \frac{120(b-a)^5}{80n^4} \leq 0.001$$

$$\frac{120(1-0)^5}{80n^4} \leq 0.001$$

$$n^4 \geq 1500$$

$$n \geq 6.22$$

$$\mathbf{n > 7}$$